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On Euclidean connections for su(1,1), su$_q$(1,1) and the algebraic approach to scattering

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Abstract. We obtain a general Euclidean connection for the su(1,1) and su$_q$(1,1) algebras. Our Euclidean connection allows an algebraic derivation for the S-matrix. These algebraic S-matrices reduce to the known ones in suitable circumstances. We also obtain a map between the su(1,1) and su$_q$(1,1) representations.

The Euclidean connection for the su(1,1) Lie algebra, i.e. a realization of the su(1,1) operators in terms of the e(2) Lie algebra operators, allows an algebraic determination of the S-matrix [1]. The q-deformation of the su(1,1) Lie algebra, su$_q$(1,1), is an associative algebra defined by three generators J_0, $J_\pm$ and the relations [2]

$$\begin{aligned} &[J_0, J_\pm] = \pm J_\pm \\ &[J_+, J_-] = [-2J_0]_q \end{aligned} \tag{1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad \text{or} \qquad [x]_q = \frac{\sinh \tau x}{\sinh \tau}$$

with $q = e^\tau$ and we consider only the cases $q \in \mathbb{R}$ or $q \in \mathbb{C}$, $|q| = 1$. Relations (1) can be seen as a deformation of the su(1,1) Lie algebra commutators. These relations reduce to the su(1,1) ones for $q \to 1$. Very recently Alonso and Frank [5] used su$_q$(1,1) to obtain an algebraic S-matrix. They prove that the operators

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= \frac{p_+}{k}\left([j_0 + \tfrac{1}{2}]_q + [\mathrm{i}\eta]_q\right) \\ J_- &= \frac{p_-}{k}\left([j_0 - \tfrac{1}{2}]_q - [\mathrm{i}\eta]_q\right) \end{aligned} \tag{2}$$

are a representation of su$_q$(1,1) with $C_q = [\frac{1}{2}]_q^2 - [\mathrm{i}\eta]_q^2$ where $j = -\frac{1}{2} + \mathrm{i}\eta$ provided that the operators j_0, $p_\pm$ are a representation of e(2) with $p_+p_- = k^2$. The relations (2) give the following recurrence relation for the S-matrix

$$S_{m+1}^q(k) = \frac{[m + \frac{1}{2}]_q + [\mathrm{i}\eta_+(k)]_q}{[m + \frac{1}{2}]_q + [\mathrm{i}\eta_-(k)]_q} S_m^q(k). \tag{3}$$

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The dependence $[\mathrm{i}\eta_\pm(k)]_q$ can be obtained if the Hamiltonian is taken as a function of C_q. We note that the sign of $\mathrm{i}\eta(k)$ cannot be fixed by algebraic techniques. Relation (3) reduces to

$$S_{m+1}(k) = \frac{m + \frac{1}{2} + \mathrm{i}\eta_+(k)}{m + \frac{1}{2} + \mathrm{i}\eta_-(k)} S_m(k) \tag{4}$$

when $q \to 1$. It is straightforward to prove that another Euclidean connection can be written:

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= \frac{p_+}{k}[j_0 + \tfrac{1}{2} + \mathrm{i}\eta]_q \\ J_- &= \frac{p_-}{k}[j_0 - \tfrac{1}{2} - \mathrm{i}\eta]_q \end{aligned} \tag{5}$$

and $C_q = [\frac{1}{2}]_q^2 - [\mathrm{i}\eta]_q^2$. These two possibilities for writing the Euclidean connection suggest splitting η into two q-terms in an arbitrary way. Thus, we could try other possible choices such as $J_0 = j_0$, $J_+ = \frac{p_+}{k}([j_0 + \frac{1}{2} + \mathrm{i}\eta_1]_q + [\mathrm{i}\eta_2]_q)$, $J_- = J_+^\dagger$. Among these possibilities, only the cases $\eta_1 = 0$ or $\eta_2 = 0$ (the relations (2) or (5)) fulfil the $\mathrm{su}_q(1,1)$ commutation relations.

Relations (5) give

$$S^q_{m+1}(k) = \frac{[m + \frac{1}{2} + \mathrm{i}\eta_+(k)]_q}{[m + \frac{1}{2} + \mathrm{i}\eta_-(k)]_q} S^q_m. \tag{6}$$

Relation (6) also reduces to (4) when $q \to 1$. Concerning the relation between the Hamiltonian and C_q, we can take, for example, $H = C_q - [\frac{1}{2}]_q^2$ (when $q \to 1$ we obtain the Hamiltonian of the problem with a Morse or Pöschl–Teller potential and m plays the role of the interaction strength [6]). In this case we obtain $-[\mathrm{i}\eta_\pm(k)]_q^2 = k^2$ and relation (3) reduces to

$$S^q_{m+1}(k) = \frac{[m + \frac{1}{2}]_q + \mathrm{i}k}{[m + \frac{1}{2}]_q - \mathrm{i}k} S^q_m(k) \tag{7}$$

(the signs are taken by analogy with Morse and Pöschl–Teller potentials). Then, relation (7) may be considered to describe modified Morse or Pöschl-Teller potential. For

$$H = \frac{1}{C_q - [\frac{1}{2}]_q^2}$$

(when $q \to 1$ we obtain the Hamiltonian of the two-dimensional problem with a Coulomb potential and m plays the role of the angular momentum [7]). We obtain $-[\mathrm{i}\eta_\pm(k)]_q^2 = \frac{1}{k^2}$ and (3) reduces to

$$S^q_{m+1}(k) = \frac{[m + \frac{1}{2}]_q + \frac{\mathrm{i}}{k}}{[m + \frac{1}{2}]_q - \frac{\mathrm{i}}{k}} S^q_m(k) \tag{8}$$

where the signs are taken by analogy with the Coulomb potential. Relation (8) may be considered to describe the modified Coulomb potential. In the case of our Euclidean connection (5), relation (7) must be replaced with

$$S^q_{m+1}(k) = \frac{[m+\frac{1}{2}+\mathrm{i}k]_q}{[m+\frac{1}{2}-\mathrm{i}k]_q} S^q_m(k) \tag{9}$$

and relation (8) with

$$S^q_{m+1}(k) = \frac{[m+\frac{1}{2}+\frac{\mathrm{i}}{k}]_q}{[m+\frac{1}{2}-\frac{\mathrm{i}}{k}]_q} S^q_m(k). \tag{10}$$

The above recurrence relations seem to be specific to $\mathrm{su}_q(1,1)$. In the following we shall prove that they can be obtained in the framework of the su(1,1) algebraic scattering theory. We also obtain other recurrence relations. Using $p_\pm j_0 = (j_0 \mp 1)p_\pm$ it is a straightforward exercise to prove that $p_\pm f(j_0) = f(j_0 \mp 1)p_\pm$ where f is a ratio of power series. We take f_μ as a ratio of power series with real coefficients which is an odd function in real variables. Thus, the commutator of the operators

$$J_= \frac{p_+}{k}\left\{\prod_\mu \frac{f_\mu(j_0+\frac{1}{2}+\mathrm{i}\alpha_\mu)}{f_\mu(j_0+\frac{1}{2}) \pm f_\mu(\mathrm{i}\alpha_\mu)}\right\}(j_0+\tfrac{1}{2}+\mathrm{i}\eta)$$

and

$$J_- = J_+^\dagger = \frac{p_-}{k}\left\{\prod_\mu \frac{f_\mu(j_0-\frac{1}{2}-\mathrm{i}\alpha_\mu)}{f_\mu(j_0-\frac{1}{2}) \mp f_\mu(\mathrm{i}\alpha_\mu)}\right\}(j_0-\tfrac{1}{2}-\mathrm{i}\eta)$$

where $p_+p_- = k^2$ and α_μ are real numbers, can be written as

$$[J_+, J_-] = -2j_0$$

provided that f_μ are solutions for the equation $f(x+y)f(x-y) = f^2(x) - f^2(y)$. We observe that the odd functions $f_\mu(x) = A_\mu[x]_{q_\mu}$ with $A_\mu \in \mathbb{R}$ and $q_\mu \in \mathbb{R}$ or $q_\mu \in \mathbb{C}$, $|q_\mu| = 1$ are solutions for the above equation (μ belongs to a finite set of indices). With these functions, $J_\pm$ gives the Casimir operator $C = \frac{1}{4} + \eta^2$. Therefore, we can write the Euclidean connection for su(1,1):

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= \frac{p_+}{k}\left\{\prod_\mu \frac{[j_0+\frac{1}{2}+\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0+\frac{1}{2}]_{q_\mu} \pm [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}(j_0+\tfrac{1}{2}+\mathrm{i}\eta) \\ J_- &= \frac{p_-}{k}\left\{\prod_\mu \frac{[j_0-\frac{1}{2}\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0-\frac{1}{2}]_{q_\mu} \mp [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}(j_0-\tfrac{1}{2}-\mathrm{i}\eta). \end{aligned} \tag{11}$$

The ratios of power series in j_0 which appear in the above formulae are well defined when they act on the j_0 eigenstates (the denominator is never zero as we considered the cases $q \in \mathbb{R}$ or $q \in \mathbb{C}$, $|q| = 1$). The operators (11) act on the Hilbert space H of the e(2) representation. One can consider this Hilbert space as the complex Hilbert space with the orthonormal basis

$$\{|m\rangle, m = 0, \pm 1, \pm 2, \ldots\}.$$

The operators (11) are well defined on the everywhere-dense subspace D of the Hilbert space H, consisting of finite linear combinations of the basis elements. Moreover, they transform D into D. Therefore, the product and the commutator of two operators is well defined on the subspace D. The definition of the operators only on an everywhere-dense subspace of the Hilbert space is due to the fact that the operators are unbounded [9].

Taking into account the Euclidean connection (11), we obtain the recurrence relation for the S-matrix

$$S_{m+1}(k) = \left\{ \prod_{\mu} \frac{[m + \frac{1}{2} + \mathrm{i}\alpha_{\mu}^{+}]_{q_{\mu}^{+}}}{[m + \frac{1}{2} + \mathrm{i}\alpha_{\mu}^{-}]_{q_{\mu}^{-}}} \frac{[m + \frac{1}{2}]_{q_{\mu}^{-}} \pm [\mathrm{i}\alpha_{\mu}^{-}]_{q_{\mu}^{-}}}{[m + \frac{1}{2}]_{q_{\mu}^{+}} \pm [\mathrm{i}\alpha_{\mu}^{+}]_{q_{\mu}^{+}}} \right\} \frac{m + \frac{1}{2} + \mathrm{i}\eta_{+}}{m + \frac{1}{2} + \mathrm{i}\eta_{-}} S_m(k). \tag{12}$$

The signs in the numerator and denominator of the above relation are not correlated. The relation between the Casimir operator and the Hamiltonian fixes $\eta_{\pm}^2(k)$ but the other parameters remain arbitrary. To conclude, we cannot obtain a useful recurrence relation for the S-matrix by algebraic methods. Relation (12) is too general and it cannot be simplified without other artificial assumptions. For example, if we take $\alpha_{\mu}^{+}(k) = \alpha_{\mu}^{-}(k)$, $q_{\mu}^{+}(k) = q_{\mu}^{-}(k)$, $\forall\mu$ and the same sign in the second ratio, then relation (12) reduces to

$$S_{m+1}(k) = \frac{m + \frac{1}{2} + \mathrm{i}\eta_{+}(k)}{m + \frac{1}{2} + \mathrm{i}\eta_{-}(k)} S_m(k) \tag{13}$$

which is Iachello's recurrence relation [1]. We can choose $\alpha_{\mu}^{+}(k) = \alpha_{\mu}^{-}(k)$ and $q_{\mu}^{+}(k) = q_{\mu}^{-}(k) \forall\mu \neq \mu_0$ and $\alpha_{\mu_0}^{\pm}(k) = \alpha^{\pm}(k)$, $q_{\mu_0}^{\pm}(k) = q$ and (12) reduces to

$$S_{m+1}(k) = \frac{[m + \frac{1}{2} + \mathrm{i}\alpha^{+}(k)]_q}{[m + \frac{1}{2} + \mathrm{i}\alpha^{-}(k)]_q} \frac{[m + \frac{1}{2}]_q \pm [\mathrm{i}\alpha^{-}(k)]_q}{[m + \frac{1}{2}]_q \pm [\mathrm{i}\alpha^{+}(k)]_q} S_m(k) \tag{14}$$

if $\eta_{+}(k) = \eta_{-}(k)$. A suitable particular case is $\alpha^{+}(k) = \alpha(k)$, $\alpha^{-}(k) = -\alpha(k)$ and the recurrence relation (14) with different signs in the second ratio gives us

$$S_{m+1}(k) = \frac{[m + \frac{1}{2} + \mathrm{i}\alpha(k)]_q}{[m + \frac{1}{2} - \mathrm{i}\alpha(k)]_q} S_m(k)$$

which is identical to (6) with $\eta_{+}(k) = -\eta_{-}(k) = \alpha(k)$. With $\alpha_{+}(k) = \alpha(k)$, $\alpha_{-}(k) = \alpha(k)$ and different signs in the second ratio ($-$ in the denominator) we obtain

$$S_{m+1}(k) = \frac{[m + \frac{1}{2}]_q + [\mathrm{i}\alpha(k)]_q}{[m + \frac{1}{2}]_q - [\mathrm{i}\alpha(k)]_q} S_m(k)$$

which is identical to (3) with $\eta_{+}(k) = -\eta_{-}(k) = \alpha(k)$. We note that if $q_{\mu}^{+}(k) = q_{\mu}^{-}(k) = 1$, $\forall\mu$, then we obtain from the relation (12)

$$S_{m+1}(k) = \left\{ \prod_{\mu} \frac{m + \frac{1}{2} + \mathrm{i}\alpha_{\mu}^{+}}{m + \frac{1}{2} \pm \mathrm{i}\alpha_{\mu}^{+}} \frac{m + \frac{1}{2} \pm \mathrm{i}\alpha_{\mu}^{-}}{m + \frac{1}{2} + \mathrm{i}\alpha_{\mu}^{-}} \right\} \frac{m + \frac{1}{2} + \mathrm{i}\eta_{+}}{m + \frac{1}{2} + \mathrm{i}\eta_{-}} S_m(k)$$

which contains as a particular case

$$S_{m+1}(k) = \left[\frac{m+\frac{1}{2}+\mathrm{i}\eta(k)}{m+\frac{1}{2}-\mathrm{i}\eta(k)}\right]^{n} S_m(k). \tag{15}$$

The above relation yields an S-matrix which is a product of Γ function ratios. In the same manner we can write the Euclidean connections for $\mathrm{su}_q(1,1)$:

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= \frac{p_+}{k}\left\{\prod \mu \frac{[j_0+\frac{1}{2}+\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0+\frac{1}{2}]_{q_\mu} \pm [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}[j_0+\tfrac{1}{2}+\mathrm{i}\eta]_q \\ J_- &= \frac{p_-}{k}\left\{\prod_\mu \frac{[j_0-\frac{1}{2}-\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0-\frac{1}{2}]_{q_\mu} \mp [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}[j_0-\tfrac{1}{2}-\mathrm{i}\eta]_q \end{aligned} \tag{16}$$

or

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= \frac{p_+}{k}\left\{\prod_\mu \frac{[j_0+\frac{1}{2}+\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0+\frac{1}{2}]_{q_\mu} \pm [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}\left([j_0+\tfrac{1}{2}]_q + [\mathrm{i}\eta]_q\right) \\ J_- &= \frac{p_-}{k}\left\{\prod_\mu \frac{[j_0-\frac{1}{2}-\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0-\frac{1}{2}]_{q_\mu} \mp [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}\left([j_0-\tfrac{1}{2}]_q - [\mathrm{i}\eta]_q\right). \end{aligned} \tag{17}$$

The recurrence relations obtained using (16) or (17) are the same as (12). We observe that (16) together with (11) yield a map between the representation of su(1,1) with $C = \frac{1}{4}+\eta^2$ and the representation of $\mathrm{su}_q(1,1)$ with $C = [\frac{1}{2}]_q - [\mathrm{i}\xi]_q^2$:

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= j_+\left\{\prod_\mu \frac{[j_0+\frac{1}{2}+\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0+\frac{1}{2}]_{q_\mu} \pm [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}\left\{\prod_\nu \frac{[j_0+\frac{1}{2}]_{q_\nu} \pm [\mathrm{i}\alpha_\nu]_{q_\nu}}{[j_0+\frac{1}{2}+\mathrm{i}\alpha_\nu]_{q_\nu}}\right\}\frac{[j_0+\frac{1}{2}+\mathrm{i}\xi]_q}{j_0+\frac{1}{2}+\mathrm{i}\eta} \\ J_- &= j_-\left\{\prod_\mu \frac{[j_0-\frac{1}{2}-\mathrm{i}\alpha_\mu]_{q_\mu}}{[j_0-\frac{1}{2}]_{q_\mu} \mp [\mathrm{i}\alpha_\mu]_{q_\mu}}\right\}\left\{\prod_\nu \frac{[j_0-\frac{1}{2}]_{q_\nu} \mp [\mathrm{i}\alpha_\nu]_{q_\nu}}{[j_0-\frac{1}{2}-\mathrm{i}\alpha_\nu]_{q_\nu}}\right\}\frac{[j_0-\frac{1}{2}-\mathrm{i}\xi]_q}{j_0-\frac{1}{2}-\mathrm{i}\eta}. \end{aligned} \tag{18}$$

The above relations contain as a particular case

$$\begin{aligned} J_0 &= j_0 \\ J_+ &= j_+\frac{[j_0+\frac{1}{2}+\mathrm{i}\xi]_q}{j_0+\frac{1}{2}+\mathrm{i}\eta} \\ J_- &= j_-\frac{[j_0-\frac{1}{2}-\mathrm{i}\xi]_q}{j_0-\frac{1}{2}-\mathrm{i}\eta}j_- \end{aligned} \tag{19}$$

(compare to [8]).

In conclusion we have obtained for the first time a general Euclidean connection for su(1,1) and $su_q(1,1)$. We have also obtained a map between the su(1,1) and $su_q(1,1)$ representations. In suitable circumstances our Euclidean connections reduce to the known ones.

These Euclidean connections yield S-matrix recurrence relations which generalize the known ones. Moreover, the same recurrence relations can be obtained using su(1,1) or $su_q(1,1)$ if one takes the appropriate Euclidean connections. We emphasize that there is no algebraic method to choose a particular Euclidean connection.

References

[1] Alhassid Y, Gursey F and Iachello F 1986 *Ann. Phys., NY* **167** 181

[2] Drinfeld V G 1985 *Sov. Math. Dokl.* **32** 254; 1988 *Sov. Math. Dokl.* **36** 212
Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 243
Woronowicz S 1987 *Commun. Math. Phys.* **111** 613

[3] Gupta R K and Ludu A 1993 *Phys. Rev.* C **48** 593

[4] Bonatsos D and Daskaloyanis C 1993 *Phys. Rev.* A **48** 3611

[5] Alonso C E and Frank A 1993 unpublished
Frank A, Alonso C E and Gomez-Camacho J 1993 *Rev. Mex. Fis.* **39** 64

[6] Frank A and Wolf K B 1984 *Phys. Rev. Lett.* **52** 1737

[7] Alhassid Y, Engel J and Wu J 1984 *Phys. Rev. Lett.* **53** 17

[8] Curtright T L and Zachos C K 1990 *Phys. Lett.* **243B** 237

[9] Riesz F and Sz-Nagy B 1956 *Funct. Anal.*